Math 2450: Lines and Planes in $\mathbb{R}^3$

What is a line in $\mathbb{R}^3$? Previously, we explored lines in two-dimensions ($\mathbb{R}^2$), using a point on the line (x_1, y_1) and slope $m = \frac{b}{a}$. Recall the slope-intercept form of a line:

$$y = mx + b$$

We can think of slope as a direction $\langle a, b \rangle$, where a is the change in x and b is the change in y. The direction can be represented as the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. But how do we express lines in three-dimensions ($\mathbb{R}^3$)? We now have 3 variables (x, y, z), so our direction vector is of the form $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Below is the formal definition of the parametric form of a line in $\mathbb{R}^3$:

Parametric Form of a Line in $\mathbb{R}^3$: If L is a line that contains the point (x_0, y_0, z_0) and is aligned with $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the point (x, y, z) is on L if and only if its coordinates satisfy:

$$x - x_0 = tA \qquad y - y_0 = tB \qquad z - z_0 = tC$$

or

$$\langle x, y, z \rangle = \langle x_0 + At, y_0 + Bt, z_0 + Ct \rangle$$

If $A, B, C \neq 0$, we may rearrange the equations above to obtain the **symmetric equations** for a line:

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

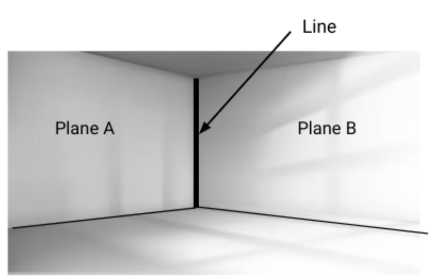
Note: In this case, $\langle A, B, C \rangle$ is parallel with the line.

What is a plane in $\mathbb{R}^3$? A plane is a two-dimensional surface in three-dimensional space. The equation of a plane may be expressed two different ways:

- **Point-normal form:** $A(x - x_0) + B(y - y_0) + C(z - z_0) + D = 0$
- **Standard form:** $Ax + By + Cz + D = 0$ for some constants A, B, C, D

In the point-normal form of a plane, (x_0, y_0, z_0) is a point contained in the plane and $\mathbf{N} = \langle A, B, C \rangle$ is the normal vector that is orthogonal to every vector in the plane. Remember that orthogonal vectors have a dot product of 0. Standard form is a simplification of point-normal form.

Why are lines and planes in $\mathbb{R}^3$ important? Lines and planes in $\mathbb{R}^3$ are the foundation for working with all other surfaces and shapes in three-dimensional space. Planes are two-dimensional and contain infinitely many lines, whereas lines are one-dimensional and contain infinitely many points. Lines in $\mathbb{R}^2$ ($y = mx + b$) describe planes in $\mathbb{R}^3$, therefore, it is important to understand the definition of a line specific to $\mathbb{R}^3$. Consider the walls of a room:



Note that the room is made up of planes. The intersection of the planes creates a line. These are the building blocks of three-dimensional shapes that we will explore throughout the course.

Example 1. Find the parametric equations for the line that contains the point $(1, 2, 3)$ and is aligned with the vector $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. Find where this line passes through the coordinate planes.

Solution: From the given vector $\mathbf{v}$, we see that $A = 2$, $B = 1$, and $C = -1$. Since the point $(1, 2, 3)$ is contained on the line, we have that $x_0 = 1$, $y_0 = 2$, $z_0 = 3$.
Now using the definition of the parametric form of a line in $\mathbb{R}^3$ (provided above), we get the line:

$$\boxed{\begin{array}{ccc} x - 1 = 2t & y - 2 = 1t & z - 3 = -1t \\ x = 1 + 2t & y = 2 + t & z = 3 - t \end{array}}$$

To find where our line passes through (or intersects) the coordinate planes, we first need to understand what it means for a line in $\mathbb{R}^3$ to intersect a plane. Consider the xy-plane: it is two-dimensional and does not have a z-coordinate. Therefore, in the xy-plane, $z = 0$. So, for our line, when we let $z = 0$:

$$\begin{array}{cccc} z = 3 - t & \rightarrow & 0 = 3 - t & [z = 0] \\ & \rightarrow & t = 3 & \end{array}$$

We get that the value $t = 3$. Using this we find the values of x and y at the point of intersection with the xy-plane:

$$\begin{array}{cccc|cccc} x = 1 + 2t & \rightarrow & x = 1 + 2(3) & [t = 3] & y = 2 + t & \rightarrow & y = 2 + 3 & [t = 3] \\ & & x = 7 & & & & y = 5 & \end{array}$$

The point of intersection is $(7, 5, 0)$. A similar process is followed to find the intersection of our line with the yz-plane (set $x = 0$) and xz-plane (set $y = 0$). Check the points of intersection of the line:

- with the yz-plane: $(0, \frac{3}{2}, \frac{5}{2})$
- with the xz-plane:$(-3, 0, 5)$

Example 2. Find a vector $\mathbf{v}$ in the same direction of the line given below:

$$x = 6 - 2t \qquad y = 1 + t \qquad z = 3t$$

Solution: Using the definition of the parametric form of a line in $\mathbb{R}^3$, the vector $\langle A, B, C \rangle$ that is parallel to the given line is the coefficients of the t terms:

$$A = -2 \qquad B = 1 \qquad C = 3$$

Therefore, a vector in the same direction of the given line is:

$$\mathbf{v} = \langle -2, 1, 3 \rangle$$

Example 3. Find normal vectors to the planes

1. $-x + 4y = 3$ [**Solution:** $\mathbf{N} = \langle -1, 4, 0 \rangle$]
2. $0.6x + y - 2.3z = 10$ [**Solution:** $\mathbf{N} = \langle 0.6, 1, -2.3 \rangle$]
3. $3y - 2z = 1$ [**Solution:** $\mathbf{N} = \langle 0, 3, -2 \rangle$]

Practice Problems

1. Find the parametric equations for the line that contains the point $(-2, 3, 5)$ and is aligned with the vector $\mathbf{v} = \langle 4, -1, 7 \rangle$. Find where the line intersects the xz-plane.

   [**Solution:** Line: $x = -2 + 4t \quad y = 3 - t \quad z = 5 + 7t$ and point of intersection: $(10, 0, 26)$]

2. Find a vector in the same direction of the line: $\langle 5 + t, 3 - 7t, 2 - 4t \rangle$

   [**Solution:** $\mathbf{v} = 1\mathbf{i} - 7\mathbf{j} - 4\mathbf{k}$]